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A Mathematical Theory and Applications of Non-Manifold Geometric Modeling

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We present a non-manifold geometric modeling system that can manipulate wireframe, surface, and solid models in a single architecture. We first introduce a mathematical framework for characterizing non-manifold geometric models, and then discuss the characteristics of non-manifold geometric modeling on the basis of the mathematical framework; in particular, we propose new Euler operations that are suitable for manipulating topological data.

To prove the effectiveness of non-manifold geometric models for improving of a 3D modeling environment, we present a rapid and flexible method of reshaping solid models allowed by hybrid CSG/B-rep modeling, based on non-manifold geometric modeling.

Keywords: CAD, geometric modeling, solid modeling, non-manifold, cell complex, form feature

1. Introduction

1.1 Current Geometric Modeling for Design

It is well known that 3D geometric models play an important role in mechanical CAD, and a number of application systems have been developed on the basis of such models. However, it is also recognized that they are not sufficient when whole manufacturing processes are to be supported by computerized systems. This is mainly because geometric models only represent the final shape of a product, and the meaning of the shape, which may represent the design intent, is not maintained by the system. Therefore, when necessary geometric models are used for various manufacturing applications, it is usually necessary to add some information that should have been given to the system when the model was created. In view of this, it is recognized that capturing and maintaining the design intent is very important for manufacturing applications.

The concept of *feature modeling* seems to be promising for solving these problems. In this type of modeling, a product is described in vocabulary that is familiar to designers, and therefore engineering meanings can be directly manipulated and maintained.

When geometric modeling is incorporated into feature modeling, the idea of *form features* seems to be useful. In geometric modeling by means of form features, the shape of a product is described by a set of subparts each of which has engineering meanings.

Several modeling systems have been based on the idea of form features. They have solved some problems that cannot be solved by conventional modeling systems, but fundamental problems are still left unsolved, because most existing feature-based modelers are developed by means of conventional solid modeling technologies. We believe that conventional geometric modeling has inevitable restrictions and is therefore not suitable for design, even if new ideas such as form features are introduced. The issues of conventional geometric modeling may be stated as follows:

- During design, we should be possible to represent the design intent in various forms such as wireframe, surface, and solid forms. However, current geometric models cannot represent an object composed of multiple forms with consistency in a single structure.
- In solid modeling, it is difficult to manipulate additional data that does not appear in a resultant shape. For instance, an object with cellular structures cannot be explicitly represented in a boundary representation (B-rep) solid model.
- Design should proceed by trial and error. However, current solid modeling is not suitable for this process, because it is time-consuming to cancel or modify previous modeling operations in B-rep. It is of course easy to reshape models in CSG modeling, simply because intersections of primitive objects are not explicitly represented. However, it has often been said that CSG models are only useful for certain applications. We believe that they are not suitable for design, because it is impossible to define shapes by using intersection lines and points [*Shimada89*].

1.2 Non-Manifold Geometric Modeling for Design

The idea of non-manifold geometric modeling seems to be very promising for solving these problems [*Weiler86*], because non-manifold geometric modeling can provide a method for representing wireframe, surface, and solid models, or a mixture of them in a single architecture. We think this capability will allow us the development of a new geometric modeling system that is more suitable for design [*Kawabe88*].

One of the most important properties of non-manifold geometric modeling is that the model can maintain additional data that does not appear in the resultant shape, but is meaningful for design, such as cellular structures and additional lines. A good application of this capability is hybrid B-rep/CSG representation, which has characteristics of both B-rep and CSG modeling [*Wilson86*]. This representation is realized by allowing cellular structures in a B-rep model. In this representation, a desired shape is represented by a set of geometric cells and each CSG primitive is represented by a different set of cells at the same time. Therefore, any CSG primitive can be easily extracted from the B-rep model, and consequently, the model can be reshaped quickly, which makes it suitable for modeling by trial and error.

As described above, non-manifold geometric modeling is very attractive for advanced CAD. However, a mathematical framework for discussing the characteristics of

non-manifold geometric modeling has not yet been proposed. A mathematical framework should be defined in such a way that it is possible to discuss the mathematical characteristics of non-manifold models independently of the data structures for manipulating the models in a computer.

The purposes of this paper are:

- To introduce a mathematical framework for characterizing non-manifold geometric models
- To discuss the characteristics of non-manifold geometric modeling on the basis of the mathematical framework; in particular, to propose new Euler operations that are suitable for manipulating topological data
- To propose hybrid B-rep/CSG representation that allows B-rep solid models to be reshaped independently of their construction sequence.

In this paper, we first introduce a mathematical framework for non-manifold geometric models in Section 2.1, and then discuss their topological elements in Section 2.2. In Section 2.3, we propose new Euler operations. Hybrid B-rep/CSG representation is discussed in Section 3.

2. Mathematical Theory for Non-Manifold Geometric Modeling

2.1. Mathematical Definition of Non-Manifold Geometric Models

It is known that non-manifold geometric modeling can be used to represent wireframe, surface, and solid models, or a mixture of them. It allows us to manipulate a larger variety of objects than manifold geometric modeling *Weiler86*.

As yet, however, non-manifold geometric models have not been not mathematically defined: *what objects are valid in 3D Euclidean space?* Such a mathematical definition is very important, because it would allow to study characteristics of non-manifold geometric models by using mathematical knowledge. For instance, in modeling polyhedra, we could utilize knowledge of polyhedra, such as Euler's Law.

We will define non-manifold geometric models mathematically to allow the use of mathematical theorems.

As noted by Requicha [*Requicha80*], a solid model can be defined mathematically as a subset of 3D Euclidean space E^3 that is (1) bounded, (2) closed, (3) semianalytic, and (4) regular. The set is called an *r-set*. However, the condition of regularity is not satisfied in non-manifold geometric modeling, because *regular* means that every object must be a 3D manifold.

We propose that the suitable mathematical definition of non-manifold geometric models is subsets of E^3 that are **cell complexes**. The term *cell complex* is a basic concept in the field of *topology*. Though there are several definitions of a cell complex, we restrict our discussion to a Euclidean cell complex embedded in E^3 , which is defined as a subset of E^3 . Some examples of cell complexes are shown in Figure 1.

Here we give the mathematical definition of the cell complex. First, let us introduce a few mathematical terms.

- "P and Q are **homeomorphic**" means that P can be modified into Q without cutting and pasting.
- An "**n-cell**" is a subset of 3D Euclidean space that is *homeomorphic* to an n-dimensional open sphere, defined as $\{x \mid |x| < \varepsilon, x \in E^n\}$. An n-cell is denoted by e , and dimension of e is denoted by $\dim(e)$. In a 3D Euclidean space, 0-cells, 1-cells, 2-cells, and 3-cells are used. These are shown in Figure 2.
- A "**closure**" of a cell e is a space that consists of a cell e and its own boundary. It is denoted by $[e]$.

Using these terms, the cell complex C is defined as a set of n-cells $\{e_\lambda \mid \lambda \in \Lambda\}$ that satisfies the following conditions:

1. $C = \bigcup_{\lambda \in \Lambda} e_\lambda$
2. If $\dim(e_\lambda) = n + 1$ ($\lambda \in \Lambda$), then $[e_\lambda] - e_\lambda \subset C^n$,
where $C^n = \{e_\mu \mid \dim(e_\mu) \leq n, \mu \in \Lambda\}$.
3. $e_\lambda \cap e_\mu = \phi$ ($\lambda \neq \mu$)

The mathematical notions of a cell complex may be explained intuitively by referring to the counterexamples shown in Figure 3. Case (a) has an intersection between a face and an edge, which violates the third condition. Case (b) has an edge without an end vertex, which violates the second condition.

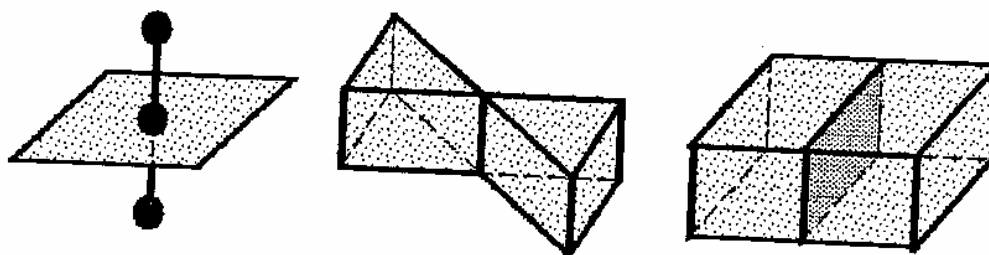


Figure 1: Examples of cell complexes.

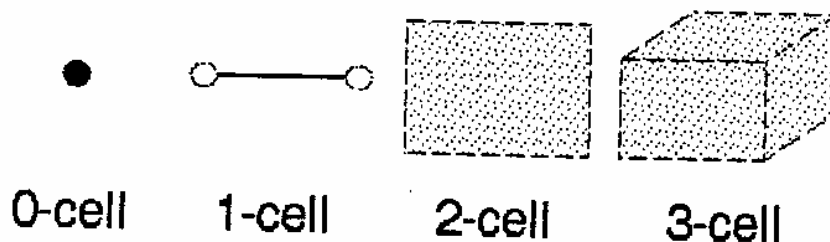


Figure 2: N-cells in 3D Euclidean space.

Overall, a non-manifold geometric model is defined as a Euclidean cell complex embedded in E^3 . Now, we can utilize mathematical theorems of the cell complex for non-manifold geometric modeling. In section 2.3, we use these theorems to define basic operations.

2.2. Topological Elements

In boundary representation (B-rep) solid modeling, an object is represented by a collection of faces, edges, and vertices. We will refer to them as *topological elements*. In addition, we regard a 3D space bounded by them as a topological element: a *volume*. A volume is defined as a 3D, continuous, bounded, open space. A topological element does not include its own boundary.

We will consider topological elements on the basis of the mathematical definition. The concept of a cell complex is similar to that of a boundary representation, because each n -cell in a cell complex is bounded by lower-dimensional n -cells.

Let us suppose that no topological element has cavities or holes. Then, each topological element can be regarded as an n -cell: a vertex is a 0-cell, an edge is a 1-cell, a face is a 2-cell, and a volume is a 3-cell. Thus, vertices, edges, faces, and volumes are sufficient to represent a non-manifold geometric model, because a cell complex is defined by a collection of 0-cells, 1-cells, 2-cells, and 3-cells.

Volumes are not manipulated explicitly in solid models, because a solid model always has just one volume. However they must be manipulated explicitly in non-manifold geometric models, which can have more than one volume. In addition, a 3D space

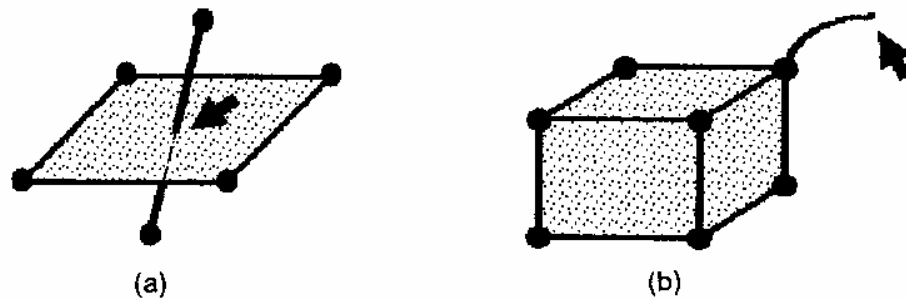


Figure 3: Examples of objects that are not cell complexes.

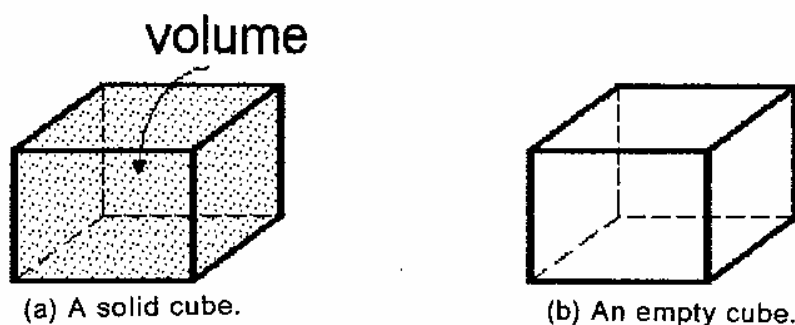


Figure 4: Two types of 3D spaces bounded by faces.

bounded by faces is not always a volume in non-manifold geometric modeling. For instance, a 3D space of a solid cube is a volume, but a 3D space of an empty cube is not a volume [Figure 4].

Now let us consider faces and volumes with cavities and holes. Mathematically speaking, such topological elements can always be divided into a set of n-cells that satisfy the conditions of a cell complex. Thus, the above discussion also applies in this case.

Topological elements with cavities have multiple disconnected boundaries. Therefore, we introduce **loops** and **shells** to represent the disconnected boundaries of faces and volumes, respectively [Figure 5].

Topological elements are hierarchically interrelated; a lower-dimensional topological element is used as the boundary of a higher-dimensional one. The hierarchical relationships of topological elements are shown in Figure 6, and an example of a boundary representation of a object, which consists of solid, surface, and wireframe forms, is shown in Figure 7.

2.3. Euler Operations for Non-Manifold Geometric Modeling

The Euler operation for conventional solid models

In manifold solid modeling, the numbers of topological elements must satisfy the Euler-Poincare formula,

$$v - e + (f - r) = 2(s - h) \quad (1)$$

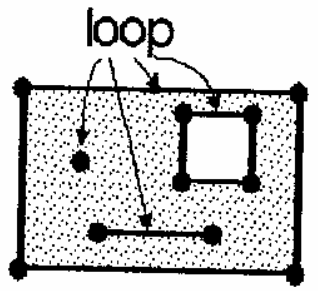
where v , e , and f are the numbers of vertices, edges, and faces, respectively, r is the number of rings that are cavities in faces, s is the number of shells that are continuous surfaces, and h is the number of holes through the object. Basic operations that generate and delete topological elements according to the Euler-Poincare formula are called Euler operations [Baumgard75]. Euler operators have the following useful features:

- A sufficient set of Euler operations can be obtained to define solid models.
- Euler operations maintain the consistency of the topological structure.
- Inverse operators exist.

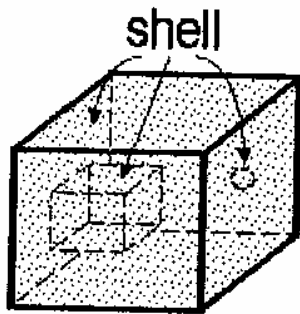
However, in non-manifold geometric modeling, formula (1) is not satisfied. For instance, in the object shown in Figure 8, $v = 7$, $e = 12$, $f = 8$, $r = 0$, $s = 1$, and $h = 0$. Hence, $v - e + (f - r) = 3$, and $2(s - h) = 2$. Thus, Euler operations for conventional solid modeling are not suitable for non-manifold geometric modeling.

In non-manifold geometric modeling, NMT operators [Weiler86] have been proposed to manipulate topological data. However they do not inherit the useful features of Euler operations, because they are not determined on the basis of a equation such as the Euler-Poincare formula. A sufficient set of NMT operations is unknown as yet.

We describe new basic operations for non-manifold geometric modeling. We call them **Euler operations** for non-manifold geometric modeling, or *extended Euler operations*,



(a) A face with cavities.



(b) A volume with cavities.

Figure 5: Representation of disconnected boundaries.

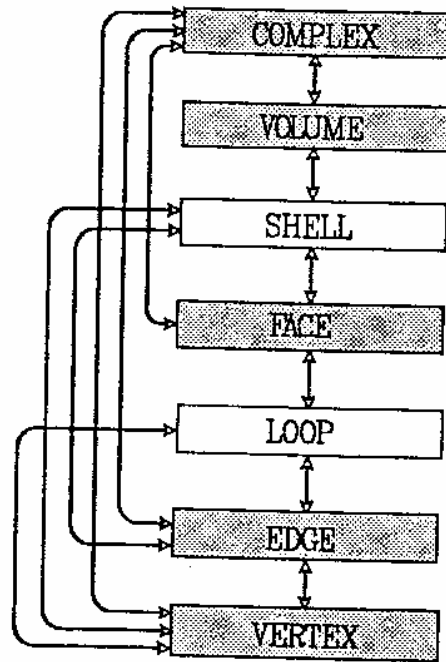


Figure 6: Hierarchical structure of topological elements.

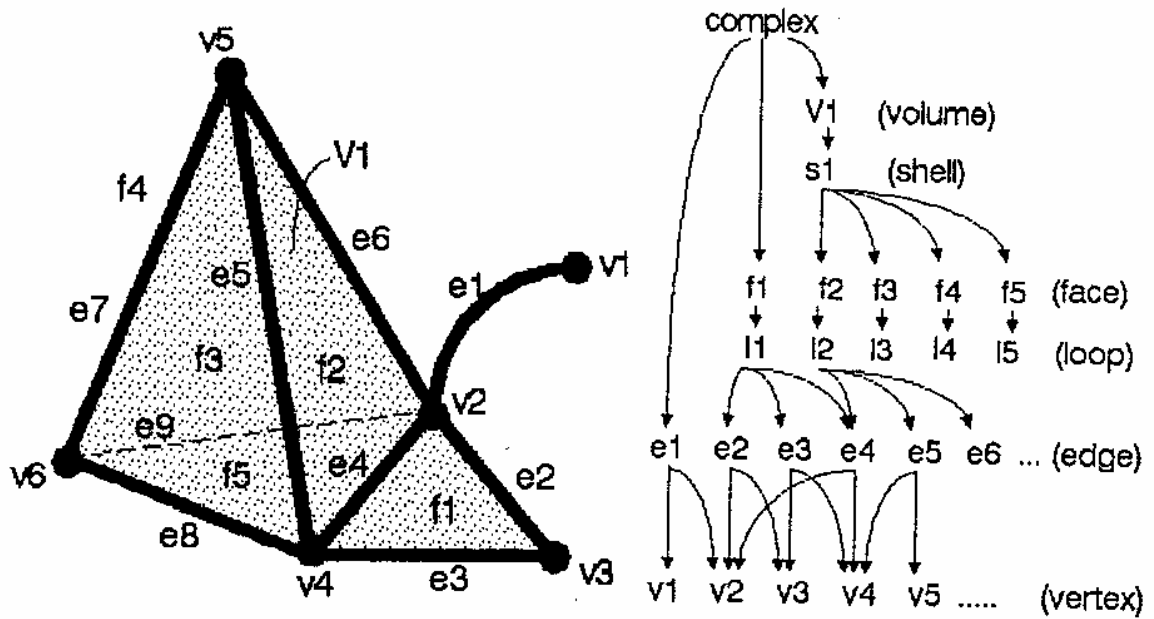


Figure 7: Representation of a pyramid with a laminar face and a dangling edge.

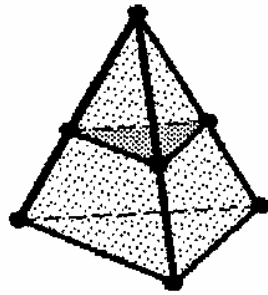


Figure 8: An object that does not satisfy the conventional Euler-Poincare formula.

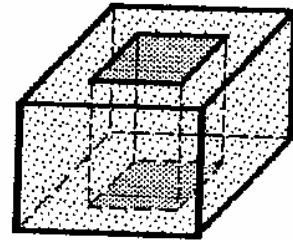
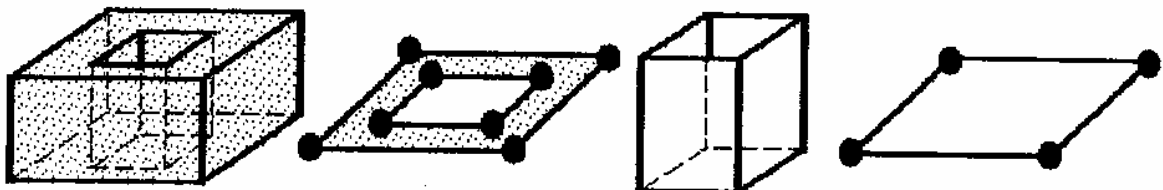


Figure 10: A solid object whose hole is filled with a volume.



(a) Holes.



(b) Cavities.

Figure 9: Examples of holes and cavities in a non-manifold geometric model.

$mVc (kVc)$	make (kill) vertex, complex	
$mev (kev)$	make (kill) edge, vertex	
$meCh (keCh)$	make (kill) edge, complex_hole	
$mFkCh (kFmCh)$	make (kill) face, kill (make) complex_hole	
$mFcc (kFcc)$	make (kill) face, complex_cavity	
$mvr (kvr)$	make vertex, ring	
$mVkc (kVmCc)$	make (kill) Volume, kill (make) complex_cavity	
$mVc (kVc)$	make (kill) vertex, Volume_cavity	
$meVh (keVh)$	make (kill) edge, Volume_hole	

Figure 11: An example of a minimum set of Euler operations.

because they are determined according to the *Euler-Poincare formula for cell complexes*.

The Euler-Poincare formula for non-manifold geometric models

In a cell complex, the numbers of n-cells must satisfy an equation, which is called the Euler-Poincare formula. Supposing that each topological element has no cavities and holes, the numbers of topological elements satisfy the following formula:

$$v - e + (f - r) = C - Ch + Cc \quad (2)$$

where v , e , and f , and V are the numbers of vertices, edges, faces and volumes, respectively. C , Ch , and Cc are the numbers of objects, holes through objects, and cavities in objects, respectively. Examples of holes and cavities are shown in Figure 9.

The formula can be generalized by adding three parameters, r , Vh , and Vc . Here r is the number of cavities in faces, and Vh is the number of holes through volumes, and Vc is the number of cavities in volumes. The formula for arbitrary non-manifold models is:

$$v - e + (f - r) + (V - Vh + Vc) = C - Ch + Cc \quad (3)$$

Equation (3) can be proved by dividing topological elements with cavities and through-holes into sets of n-cells, and applying the Euler-Poincare formula for cell complexes.

An example of the application of the Euler-Poincare formula is shown in Figure 10. In this case, there are two volumes: one has a through hole, and the other fills in the hole. The numbers of topological elements are

$$v = 16, e = 24, f = 12, r = 2, V = 2, Vh = 1, Vc = 0.$$

Since the space occupied by this object is a cube,

$$C = 1, Ch = Cc = 0.$$

The numbers of topological elements in Figure 10 satisfy the Euler-Poincare formula.

Euler operations for non-manifold geometric models

Euler operations for non-manifold geometric models are determined according to the Euler-Poincare formula (3). In non-manifold geometric modeling, nine appropriate operators (with their inverses) are sufficient to define all objects, since equation (3) expresses a plane in ten-dimensional space described by $(v, e, f, r, V, Vh, Vc, C, Ch, Cc)$, and there are nine independent base vectors. An example of a theoretical minimum set of operators is shown in Figure 11. Nine operators are sufficient to describe any object, but it is more efficient to add some other Euler operations for practical use.

An example of how a solid can be defined by using Euler operations is shown in Figure 12. It is done through wireframe, surface, and solid modeling spaces. Since a non-manifold geometric model has a larger modeling space, the choice of the sequence of Euler operations is more flexible than in manifold solid modeling.

The Euler-Poincare formula (3) is independent of data structures. In some data structures, the number of variables can be decreased. For instance, if both a volume and a volume cavity are represented by *shells* in a data structure, the minimum number of Euler operations is eight.

Euler-Poincare formulas for solid, surface, and wireframe models

Here we describe the formulas for wireframe, surface, and solid modeling. They can be derived from formula (3), because each type of model is a subset of non-manifold geometric models. Euler operations for each form can be determined on the basis of these formulas.

A solid model has just one volume, and all other topological elements are the boundary of the volume. Therefore, $V = C$, $V_h = C_h$, and $V_c = C_c$. When the number of disconnected boundaries of the volume is denoted by s , and the number of holes through the object is denoted by h , $s = C + C_c$, and $h = V_h$. Thus, the following well-known formula can be obtained:

$$v - e + (f - r) = 2(s - h) \quad (4)$$

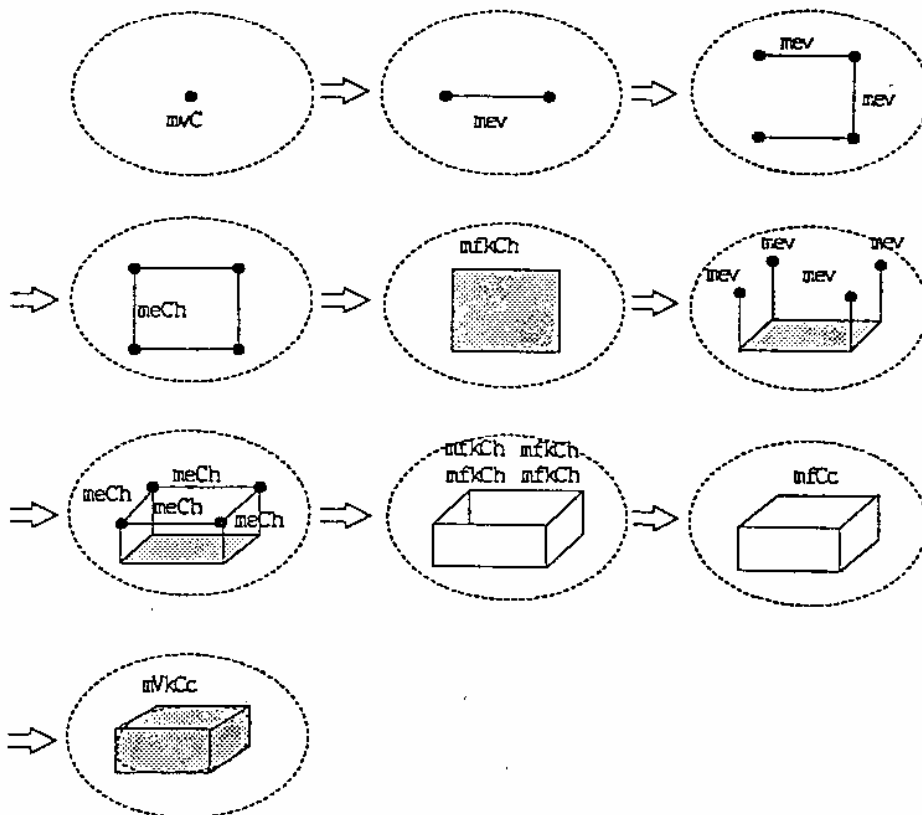


Figure 12: An example of definition of a solid by mean of the Euler operations.

In surface modeling, no volumes are defined. Therefore $V = V_h = V_c = 0$. Hence, we can obtain the formula:

$$v - e + (f - r) = C - Ch + Cc \quad (5)$$

In wireframe modeling, a object has no volumes and no faces. Hence, $f = r = V = V_h = V_c = Cc = 0$. Thus we obtain

$$v - e = C - Ch \quad (6)$$

3. Applications of Non-Manifold Geometric Modeling

We have developed a non-manifold geometric modeling system to improve a 3D solid modeling environment.

In this section, we describe one of the most promising applications of non-manifold geometric modeling: hybrid B-rep/CSG representation, which allows arbitrary reshaping of solid models defined by Boolean set operations, or some forms of local operations.

3.1 Current Status in Reshaping of 3D models

In many B-rep modelers, even when only one previously executed Boolean set operation is modified, all operations must be executed from the beginning, using the history of Boolean set operations. This has been a serious drawback in interactive modeling, because it is very time-consuming.

B-rep modelers constructed by Euler operators can undo previous operators by means of their reverse operators. In this case, it is easy to cancel a operation that has just been applied. However, to cancel a operation applied at an early stage, almost all operations must be canceled and reapplied, because the data required for the reverse operators depends on the operation sequence.

CSG models are, of course, easy to reshape, but the input data is not evaluated until they are displayed, or until corresponding B-rep models are generated. Therefore CSG models are not sufficient for some applications.

We propose a new method that allows B-rep solid models to be reshaped independently of the operation sequence.

3.2 Hybrid B-rep/CSG Representation

Non-manifold geometric models can maintain additional data, which may not appear in the resultant shape. This is one of their most useful characteristics, as it allows hybrid representation.

Hybrid representation is a modeling form that has characteristics of both CSG modeling and B-rep modeling. CSG and B-rep have complementary merits and

demerits: CSG models maintain operations and primitives, but they are not evaluated. B-rep models are evaluated, but they maintain only resultant shapes, and primitives do not remain in B-rep models.

The concept of hybrid representation in non-manifold geometric modeling is as follows. In conventional modeling, when a Boolean set operation is applied to two primitive objects, only the resultant shape is left. For example, in the union operation, all topological elements of one object that are inside the other object must be deleted. However, in non-manifold geometric modeling, such deleted topological elements can be maintained consistently [Figure 13(a)(b)].

The resultant shape is represented by adding a mark (VALID or VOID) to every topological element of a non-manifold geometric model. In the case of solid objects, the marks of volumes can determine the marks of all other topological elements. They are determined as follows.

1. VALID faces belong to only one VALID volume.
2. VALID edges belong to VALID faces.
3. VALID vertices are the end points of VALID edges.

Figure 13(d)-(f) shows some examples of the resultant shapes extracted from a model with three volumes shown in Figure 13(c). A union, a difference, and an intersection of objects are represented by the same data structure, but they have different marks. The marks are renewed whenever a Boolean set operation is executed.

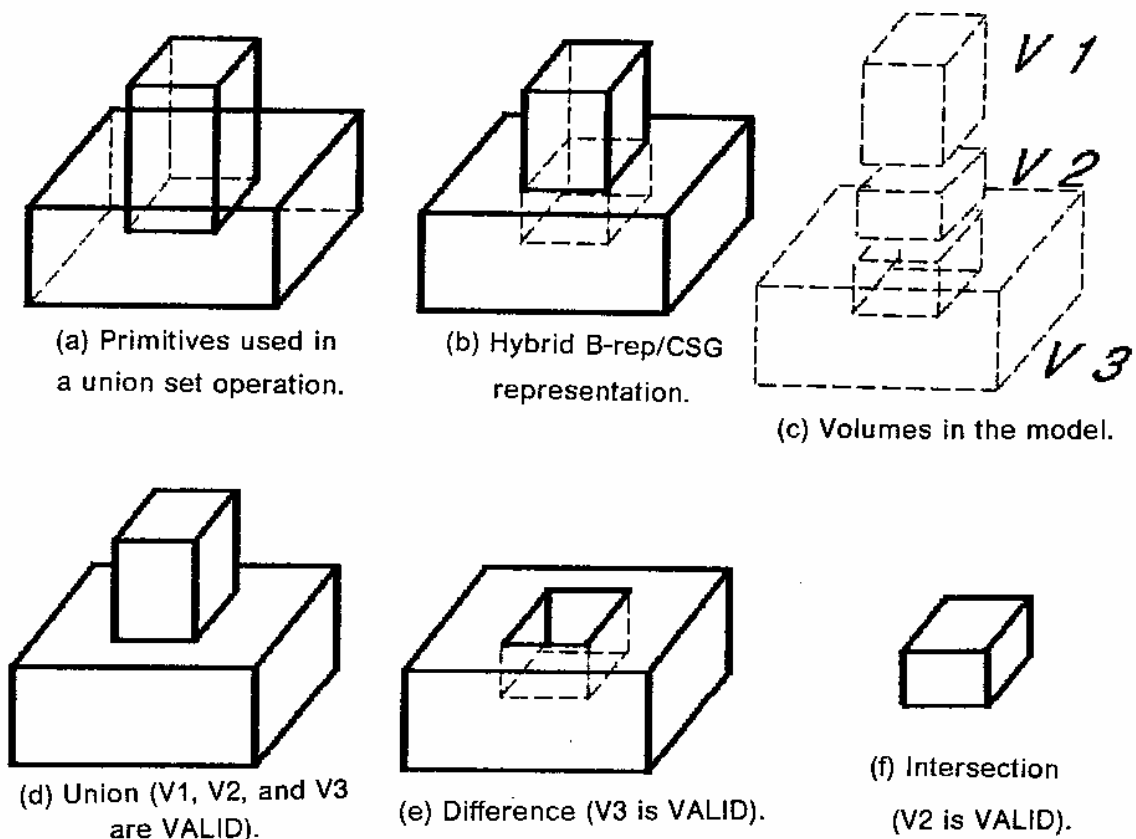


Figure 13: An example of hybrid B-rep/CSG representation.

An example of a mechanical part in hybrid B-rep/CSG representation is shown in Figure 14.

3.3 Arbitrary Reworking of Boolean Set Operations

By using this hybrid representation in non-manifold geometric models, we can realize arbitrary reworking of Boolean set operations.

A model consists of geometric primitives that are used in Boolean set operations. Since every topological element of the object model has a pointer to each primitive, we can extract each primitive from the object model.

Figure 15 shows pointers between a primitive and an object model. Every topological element of an object model is related to the topological elements of primitives. When new edges and vertices are generated by intersection, they are added to the original primitives as new topological elements. In this way, every topological element of an object model is related to topological elements of primitives.

We can cancel a Boolean set operation simply by removing the relevant primitives. This process is easy, because the system knows which topological elements must be deleted. The system deletes topological elements that are:

- related only with the deleted primitive, or
- generated by intersection between the deleted primitive and another one.

This process is done by means of the Euler operations described above, but it is not necessary to keep the sequence of the Euler operations. Any primitive can be removed and reshaped at any time.

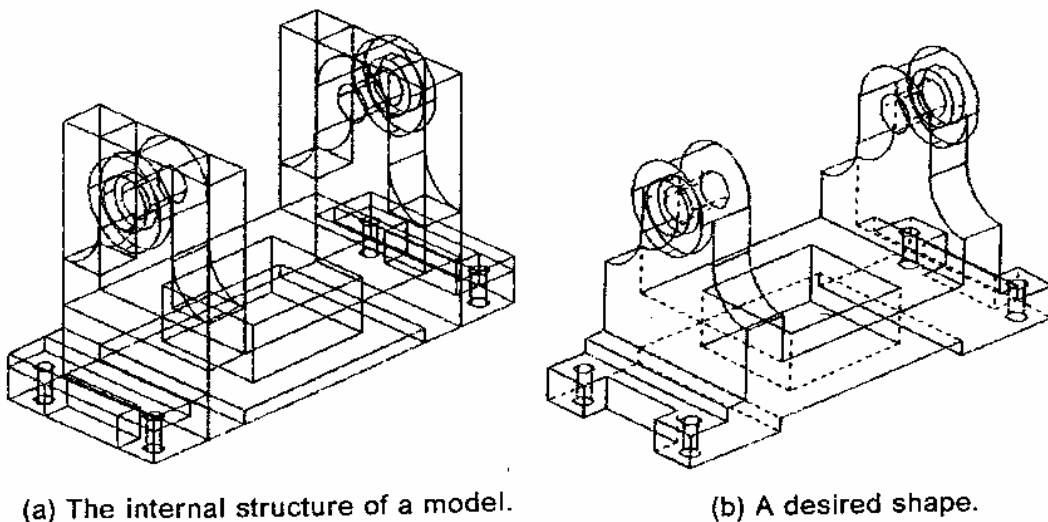


Figure 14: A hybrid B-rep/CSG model.

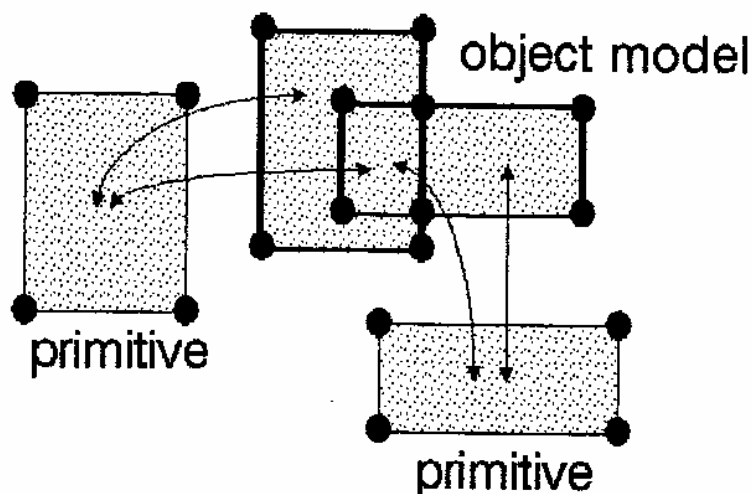


Figure 15: Relation between an object model and primitives.

4. Discussion and Conclusions

We gave a mathematical definition of non-manifold geometric models. Such a definition provides many important benefits.

Any non-manifold geometric model can be represented by a set of volumes, faces, edges, and vertices. However, theoretically sufficient topological structures to represent non-manifold geometric models defined here have not yet been derived.

New Euler operations for non-manifold geometric models were derived from the mathematical definition. The sequence of nine adequate operators can define and manipulate any non-manifold geometric model with consistency.

Hybrid B-rep/CSG representation is one of the most useful applications of non-manifold geometric modeling, and has a great influence on current solid modeling. It allows the arbitrary reshaping of a model defined by Boolean set operations.

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